Policy Gradient

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Advantages of Policy-based RL

- Previously we focused on approximating value or action-value function:

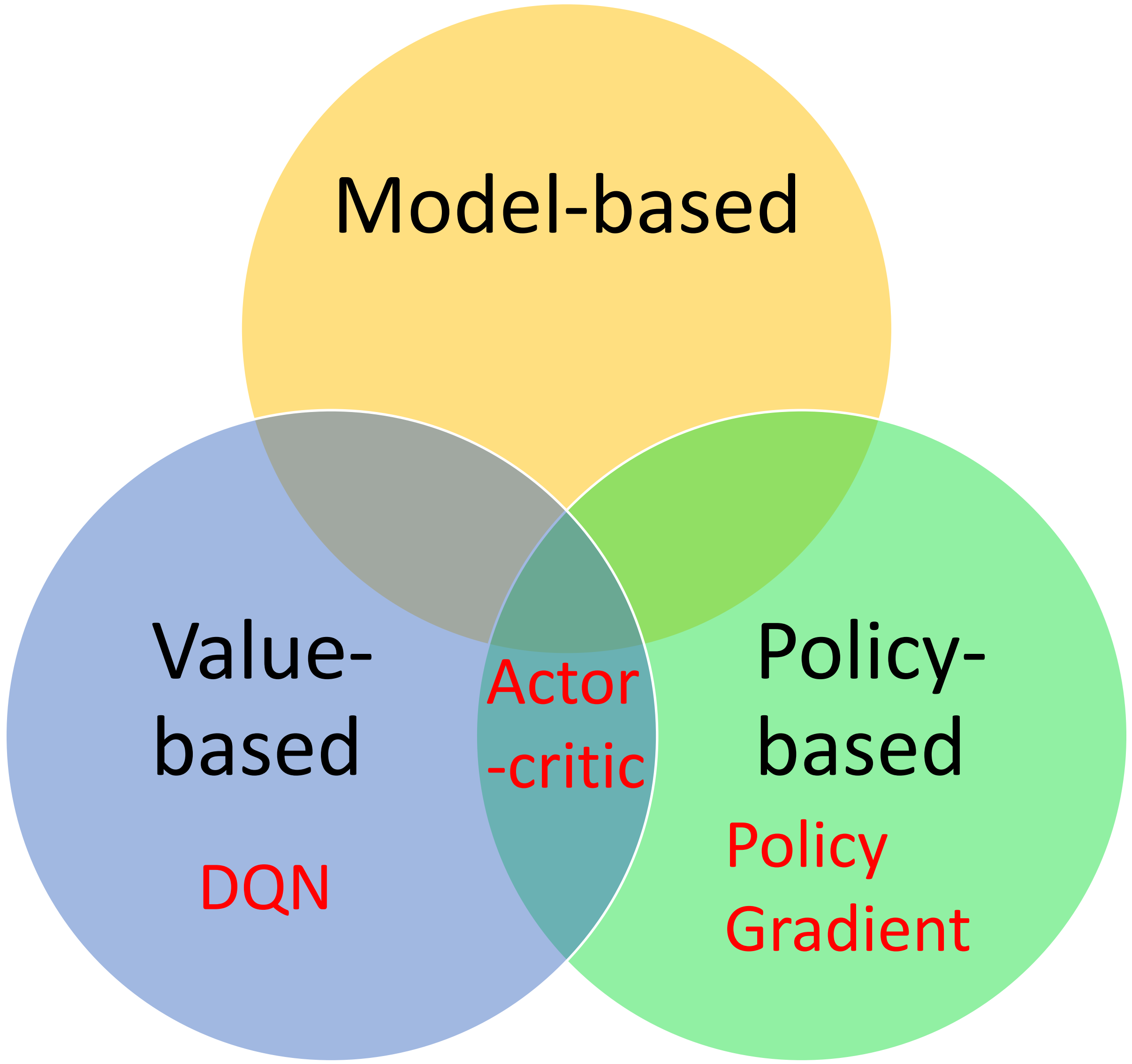
$$V_\theta(s) \approx V^\pi(s)$$
$$Q_\theta(s,a) \approx Q^\pi(s,a)$$

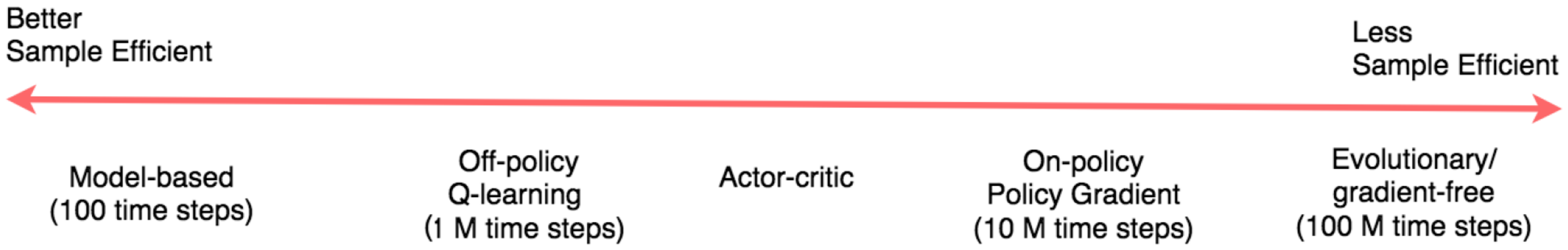
- Policy Gradient methods focus on parameterize the policy:

$$\pi_\theta(s,a) = \mathbb{P}\left[a \mid s, \theta\right]$$

3 Types of Reinforcement Learning

- Value-based
  - Learn value function
  - Implicit policy
- Policy-based
  - No value function
  - Learn Policy directly
- Actor-critic
  - Learn both value and policy function




Better Sample Efficient
Less Sample Efficient
Model-based (100 time steps)
Off-policy Q-learning (1 M time steps)
Actor-critic
On-policy Policy Gradient (10 M time steps)
Evolutionary/ gradient-free (100 M time steps)

Model-based

- Learn the model of the world, then plan using the model
- Update model often
- Re-plan often

Value-based

- Learn the state or state-action value
- Act by choosing best action in state
- Exploration is a necessary add-on

Policy-based

- Learn the stochastic policy function that maps state to action
- Act by sampling policy
- Exploration is baked in

Lex Fridman, MIT Deep Learning, https://deeplearning.mit.edu/

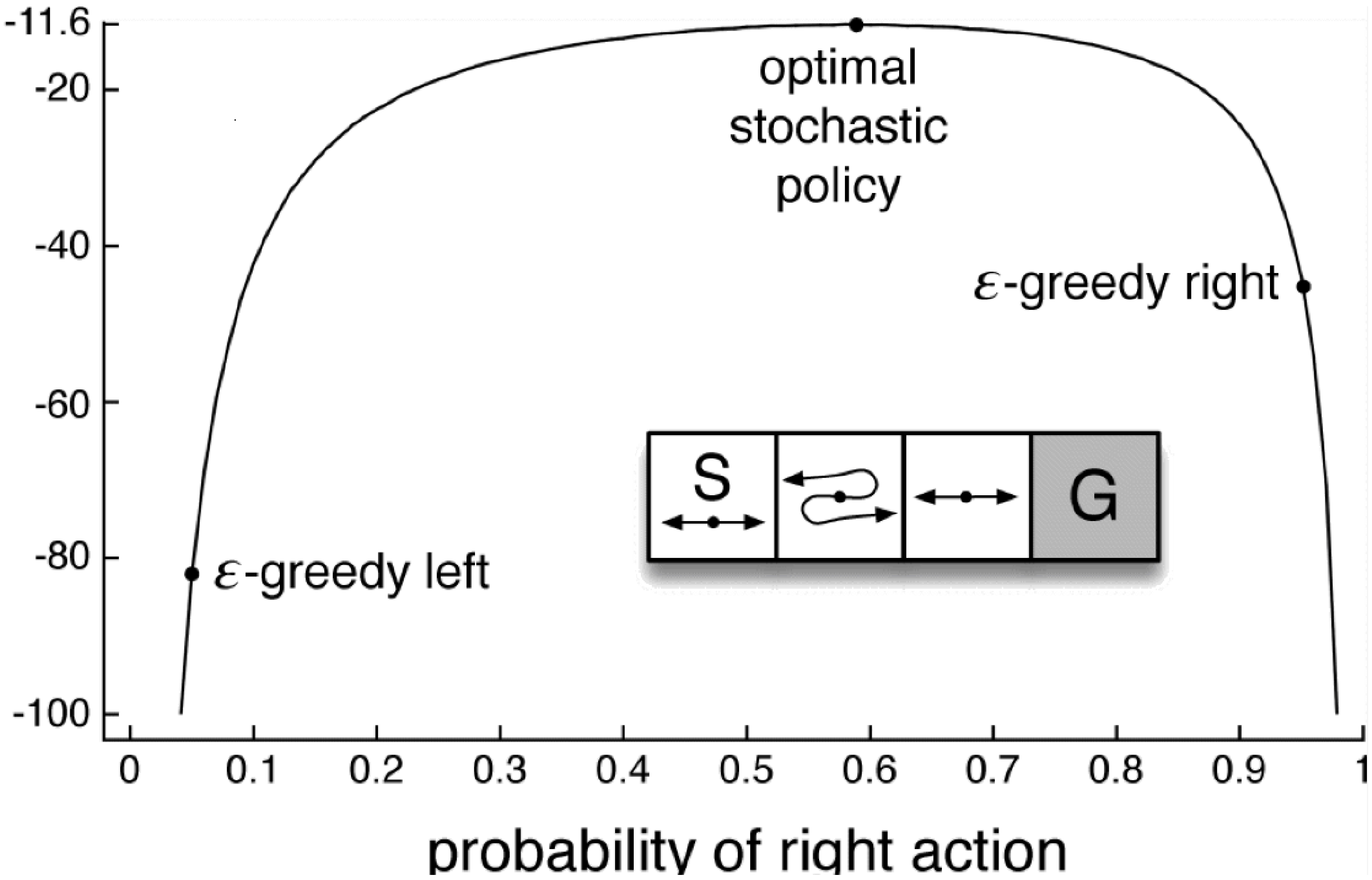
Policy Objective Function

- Goal: given policy $\pi_\theta(s, a)$ with parameters θ, find best θ
- How to measure the quality of a policy?

$$J(\theta) \leftarrow v_\pi(s_0) = E[\sum \pi(a|s) q_\pi(s, a)]$$

Short Corridor with Switched Actions

$$J(\boldsymbol{\theta}) = v_{\pi_{\boldsymbol{\theta}}}(\mathsf{S})$$



Policy Optimization

- Policy-based RL is an optimization problem that can be solved by:
  - Hill climbing
  - Simplex / amoeba / Nelder Mead
  - Genetic algorithms
  - Gradient descent
  - Conjugate gradient
  - Quasi-newton

Computing Gradients By Finite Differences

- Estimate kth partial derivative of objective function w.r.t. Θ
- By perturbing by small amount in *k*-th dimension

$$\frac{\partial J(\theta)}{\partial \theta_k} \approx \frac{J(\theta + \epsilon u_k) - J(\theta)}{\epsilon}$$

where u_k is unit vector with 1 in *k*-th component, 0 elsewhere

- Simple, noisy, inefficient but sometime work!
- Works for all kinds of policy, even if policy is not differentiable

Score Function

- Assume π_θ is differentiable whenever it is non-zero

$$\begin{aligned}\nabla_\theta \pi_\theta(s,a) &= \pi_\theta(s,a)\frac{\nabla_\theta \pi_\theta(s,a)}{\pi_\theta(s,a)} \\ &= \pi_\theta(s,a)\nabla_\theta \log \pi_\theta(s,a)\end{aligned}$$

- Score function is $\nabla_\theta \log \pi_\theta(s,a)$

Softmax Policy

- Softmax function

$$\frac{e^{\pi_\theta(s,a)}}{\sum_{i=1}^{A} e^{\pi_\theta(s,a_i)}}$$

- Use linear approximation function $\phi(s,a)^T\theta$

$$\nabla_\theta \log \pi_\theta(s,a) = \phi(s,a) - \mathbb{E}_{\pi_\theta}\left[\phi(s,\cdot)\right]$$

Policy Gradient Theorem

- Generalized policy gradient (proof @ Sutton’s book, pg.325)

$$\nabla J(\boldsymbol{\theta}) \propto \sum_s \mu(s) \sum_a q_\pi(s,a) \nabla \pi(a|s,\boldsymbol{\theta}),$$

Theorem

For any differentiable policy $\pi_\theta(s,a)$,
for any of the policy objective functions $J = J_1, J_{avR},$ *or* $\frac{1}{1-\gamma} J_{avV}$,
the policy gradient is

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\; Q^{\pi_\theta}(s,a)\right]$$

Proof of Policy Gradient Theorem (2-1)

$$\nabla v_\pi(s) = \nabla \left[\sum_a \pi(a|s) q_\pi(s,a) \right], \quad \text{for all } s \in \mathcal{S} \qquad \text{(Exercise 3.18)}$$

$$= \sum_a \Big[\nabla \pi(a|s) q_\pi(s,a) + \pi(a|s) \nabla q_\pi(s,a) \Big] \qquad \text{(product rule of calculus)}$$

$$= \sum_a \Big[\nabla \pi(a|s) q_\pi(s,a) + \pi(a|s) \nabla \sum_{s',r} p(s',r|s,a)\big(r + v_\pi(s')\big) \Big]$$

$$\text{(Exercise 3.19 and Equation 3.2)}$$

$$= \sum_a \Big[\nabla \pi(a|s) q_\pi(s,a) + \pi(a|s) \sum_{s'} p(s'|s,a) \nabla v_\pi(s') \Big] \qquad \text{(Eq. 3.4)}$$

$$= \sum_a \Big[\nabla \pi(a|s) q_\pi(s,a) + \pi(a|s) \sum_{s'} p(s'|s,a) \qquad \text{(unrolling)}$$

$$\sum_{a'} \big[\nabla \pi(a'|s') q_\pi(s',a') + \pi(a'|s') \sum_{s''} p(s''|s',a') \nabla v_\pi(s'') \big] \Big]$$

$$= \sum_{x \in \mathcal{S}} \sum_{k=0}^{\infty} \Pr(s \to x, k, \pi) \sum_a \nabla \pi(a|x) q_\pi(x,a),$$

Proof of Policy Gradient Theorem (2-1)

$$
\begin{aligned}
\nabla J(\boldsymbol{\theta}) &= \nabla v_\pi(s_0) \\
&= \sum_s \left(\sum_{k=0}^{\infty} \Pr(s_0 \to s, k, \pi) \right) \sum_a \nabla \pi(a|s) q_\pi(s,a) \\
&= \sum_s \eta(s) \sum_a \nabla \pi(a|s) q_\pi(s,a) \\
&= \sum_{s'} \eta(s') \sum_s \frac{\eta(s)}{\sum_{s'} \eta(s')} \sum_a \nabla \pi(a|s) q_\pi(s,a) \\
&= \sum_{s'} \eta(s') \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s,a) \\
&\propto \sum_s \mu(s) \sum_a \nabla \pi(a|s) q_\pi(s,a)
\end{aligned}
$$

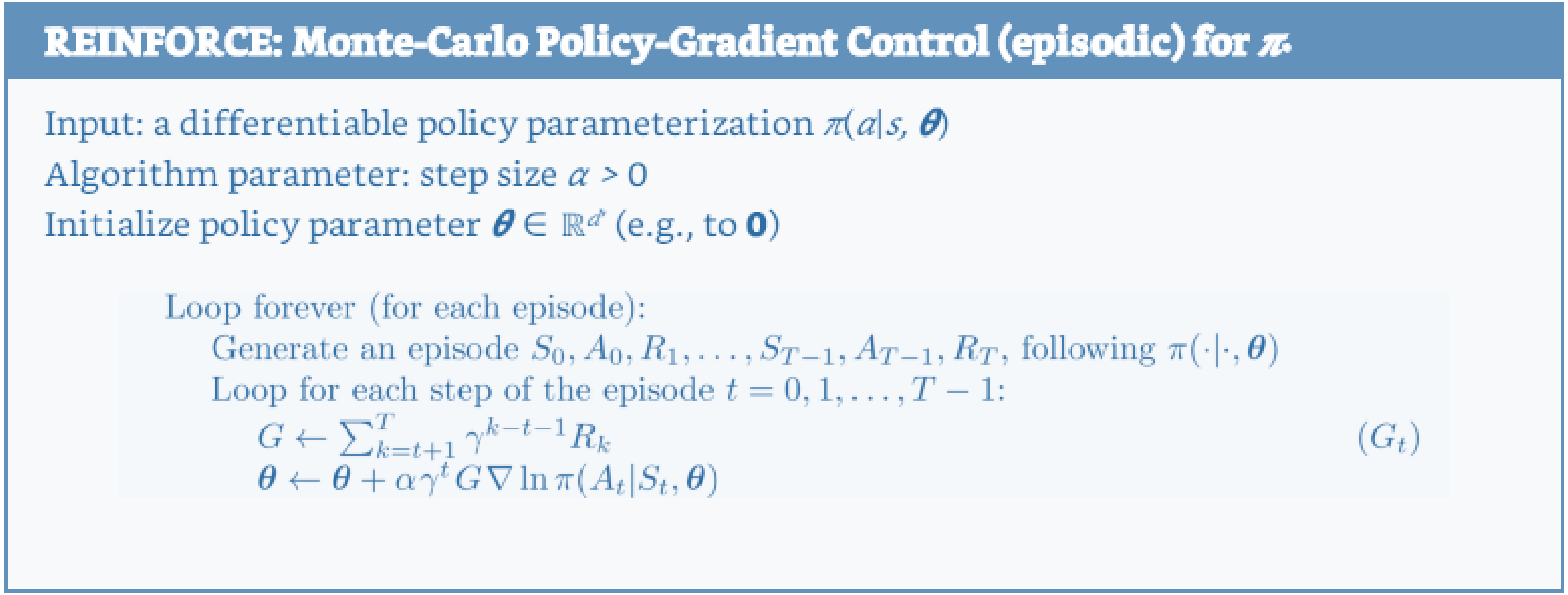
REINFOCE: Monte Carlo Policy Gradient

$$\begin{aligned}
\nabla J(\boldsymbol{\theta}) &= \mathbb{E}_\pi\left[\sum_a \pi(a|S_t,\boldsymbol{\theta})q_\pi(S_t,a)\frac{\nabla\pi(a|S_t,\boldsymbol{\theta})}{\pi(a|S_t,\boldsymbol{\theta})}\right] \\
&= \mathbb{E}_\pi\left[q_\pi(S_t,A_t)\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta})}{\pi(A_t|S_t,\boldsymbol{\theta})}\right] && \text{(replacing } a \text{ by the sample } A_t \sim \pi) \\
&= \mathbb{E}_\pi\left[G_t\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta})}{\pi(A_t|S_t,\boldsymbol{\theta})}\right], && \text{(because } \mathbb{E}_\pi[G_t|S_t,A_t] = q_\pi(S_t,A_t))
\end{aligned}$$

REINFORCE Update

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha G_t \frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)}$$

Pseudo Code of REINFORCE

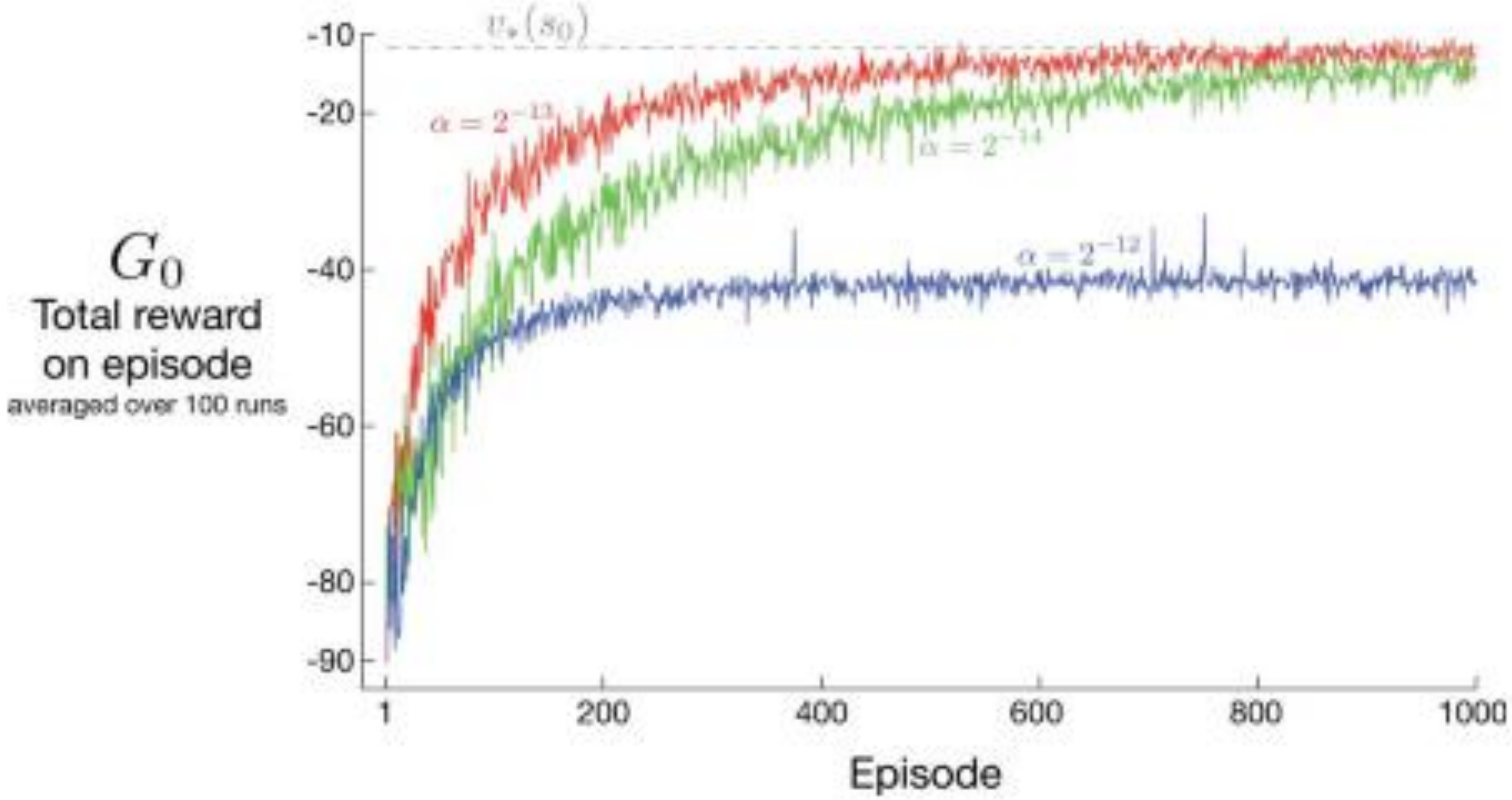


REINFORCE: Monte-Carlo Policy-Gradient Control (episodic) for π_*

```
Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$
Algorithm parameter: step size $\alpha > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):
    Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot, \boldsymbol{\theta})$
    Loop for each step of the episode $t = 0, 1, \ldots, T-1$:
        $G \leftarrow \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$                    $(G_t)$
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \gamma^t G \nabla \ln \pi(A_t|S_t, \boldsymbol{\theta})$
```

REINFORCE on Short Corridor



REINFORCE with Baseline

- Include an arbitrary baseline function b(s)

$$\nabla J(\boldsymbol{\theta}) \propto \sum_s \mu(s) \sum_a \Big(q_\pi(s,a) - b(s)\Big) \nabla \pi(a|s,\boldsymbol{\theta})$$

  – Equation is valid because

$$\sum_a b(s) \nabla \pi(a|s,\boldsymbol{\theta}) \;=\; b(s) \nabla \sum_a \pi(a|s,\boldsymbol{\theta}) \;=\; b(s) \nabla 1 \;=\; 0$$

Gradient of REINFORCE with Baseline

$$\boldsymbol{\theta}_{t+1} \doteq \boldsymbol{\theta}_t + \alpha\Big(G_t - b(S_t)\Big)\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)}$$

REINFORCE with Baseline (episodic), for estimating $\pi_\theta \approx \pi_*$

Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$

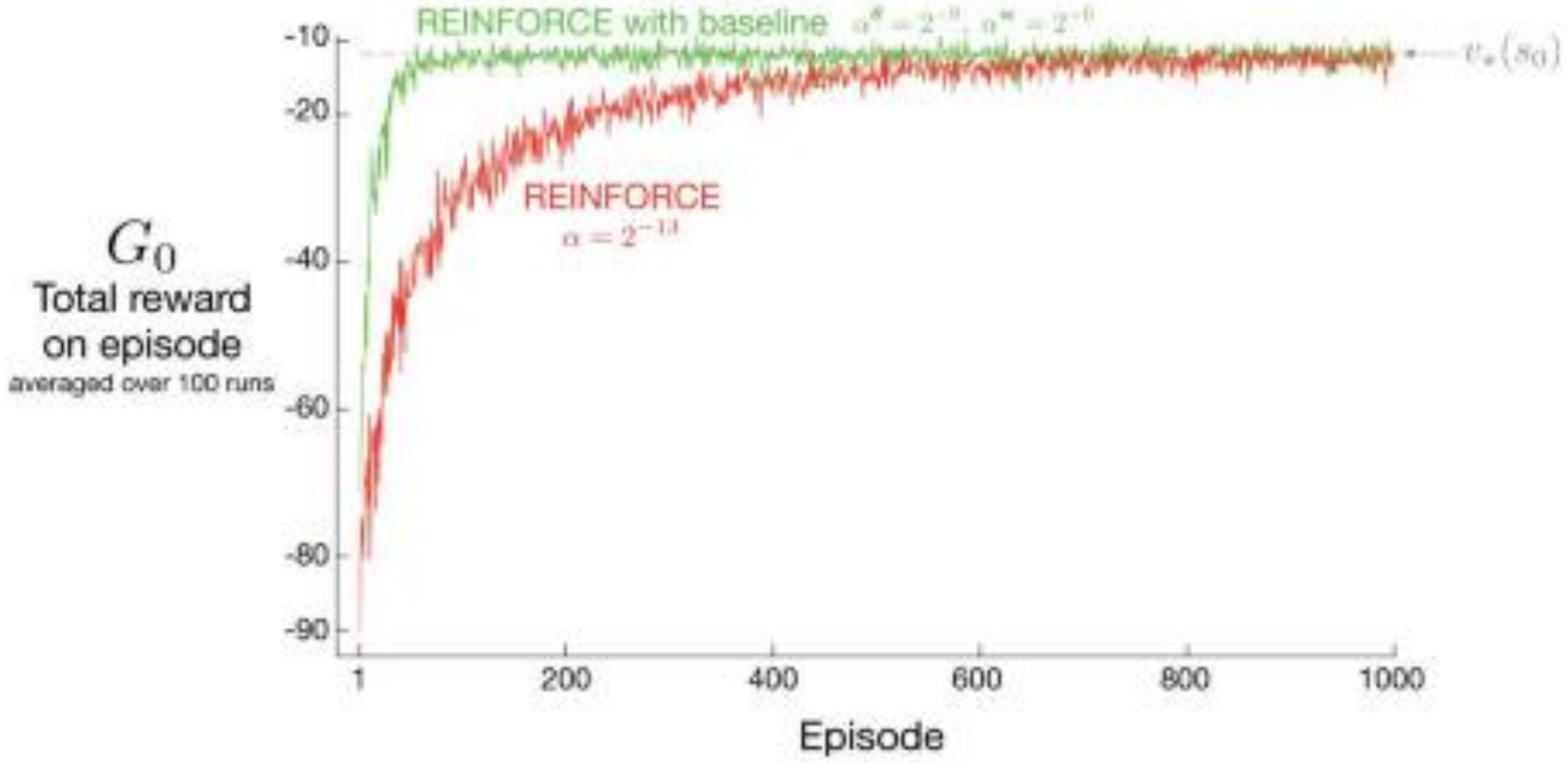
Input: a differentiable state-value function parameterization $\hat{v}(s, \mathbf{w})$

Algorithm parameters: step sizes $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\mathbf{w}} > 0$

Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ and state-value weights $\mathbf{w} \in \mathbb{R}^{d}$ (e.g., to $\mathbf{0}$)

```
Loop forever (for each episode):
    Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot,\boldsymbol{\theta})$
    Loop for each step of the episode $t = 0, 1, \ldots, T-1$:
        $G \leftarrow \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$                    $(G_t)$
        $\delta \leftarrow G - \hat{v}(S_t,\mathbf{w})$
        $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}}\, \delta \nabla \hat{v}(S_t,\mathbf{w})$
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}} \gamma^t \delta \nabla \ln \pi(A_t|S_t,\boldsymbol{\theta})$
```

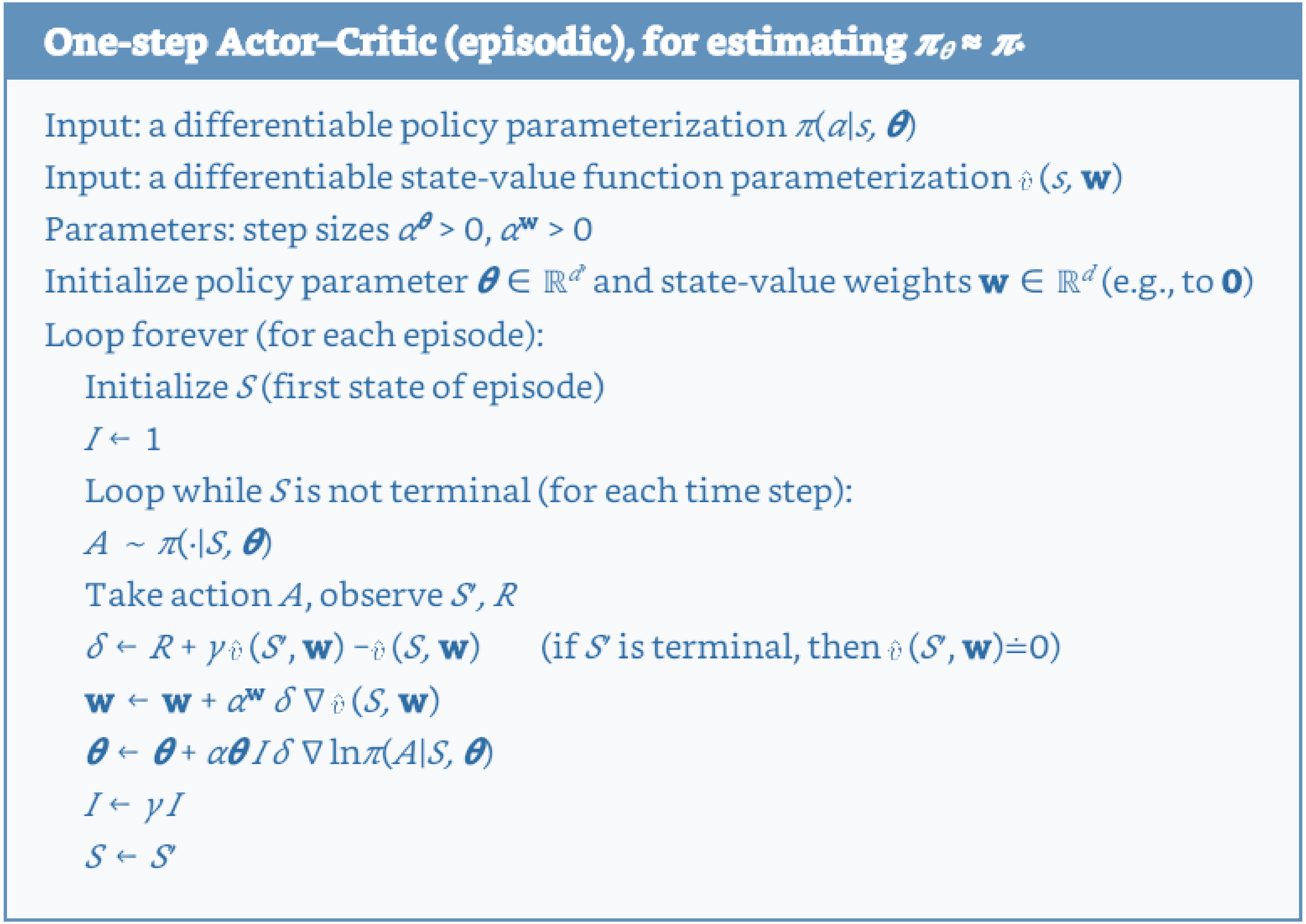
Baseline Can Help to Learn Faster



Actor-Critic Methods

- Baseline cannot bootstrap
  - Use learned state-value function as baseline -> Actor-Critic

$$\begin{aligned}
\boldsymbol{\theta}_{t+1} &\doteq \boldsymbol{\theta}_t + \alpha\Big(G_{t:t+1} - \hat{v}(S_t,\mathbf{w})\Big)\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)} \\
&= \boldsymbol{\theta}_t + \alpha\Big(R_{t+1} + \gamma\hat{v}(S_{t+1},\mathbf{w}) - \hat{v}(S_t,\mathbf{w})\Big)\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)} \\
&= \boldsymbol{\theta}_t + \alpha\,\delta_t\frac{\nabla\pi(A_t|S_t,\boldsymbol{\theta}_t)}{\pi(A_t|S_t,\boldsymbol{\theta}_t)}.
\end{aligned}$$



One-step Actor–Critic (episodic), for estimating $\pi_\theta \approx \pi_*$

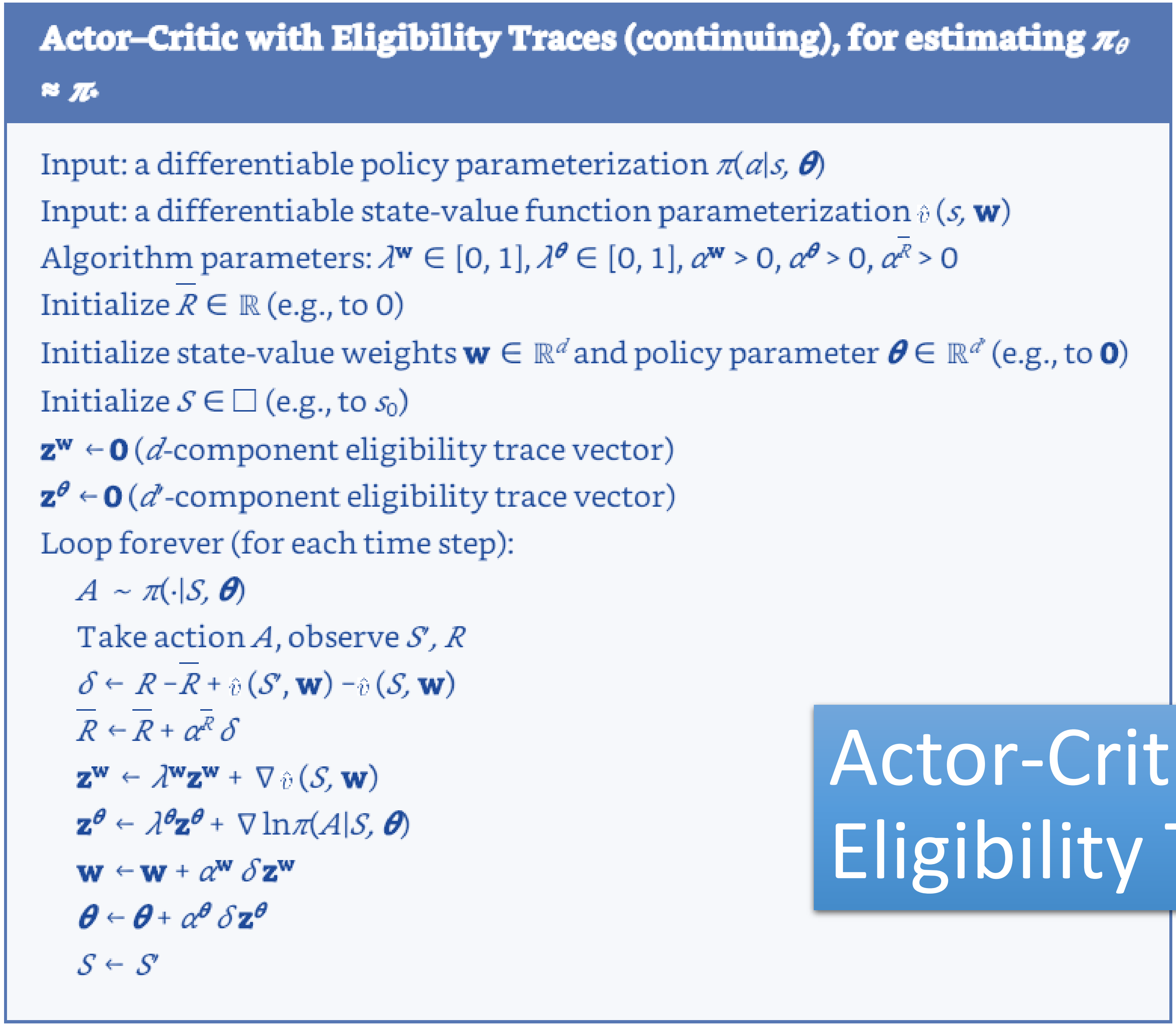
```
Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$
Input: a differentiable state-value function parameterization $\hat{v}(s, \mathbf{w})$
Parameters: step sizes $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\mathbf{w}} > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ and state-value weights $\mathbf{w} \in \mathbb{R}^{d}$ (e.g., to $\mathbf{0}$)
Loop forever (for each episode):
    Initialize $S$ (first state of episode)
    $I \leftarrow 1$
    Loop while $S$ is not terminal (for each time step):
    $A \sim \pi(\cdot|S, \boldsymbol{\theta})$
    Take action $A$, observe $S'$, $R$
    $\delta \leftarrow R + \gamma\hat{v}(S', \mathbf{w}) - \hat{v}(S, \mathbf{w})$      (if $S'$ is terminal, then $\hat{v}(S', \mathbf{w}) \doteq 0$)
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}} \delta \nabla \hat{v}(S, \mathbf{w})$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha\boldsymbol{\theta} I \delta \nabla \ln\pi(A|S, \boldsymbol{\theta})$
    $I \leftarrow \gamma I$
    $S \leftarrow S'$
```

Policy Gradient for Continuing Problems

- Continuing problem (No episode boundaries)
  - Use average reward per time step: TD(λ)

$$\begin{aligned} J(\boldsymbol{\theta}) \doteq r(\pi) &\doteq \lim_{h\to\infty} \frac{1}{h} \sum_{t=1}^{h} \mathbb{E}[R_t \mid S_0, A_{0:t-1} \sim \pi] \\ &= \lim_{t\to\infty} \mathbb{E}[R_t \mid S_0, A_{0:t-1} \sim \pi] \\ &= \sum_s \mu(s) \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) r \end{aligned}$$

Actor-Critic with Eligibility Traces



Actor–Critic with Eligibility Traces (continuing), for estimating $\pi_\theta \approx \pi_*$

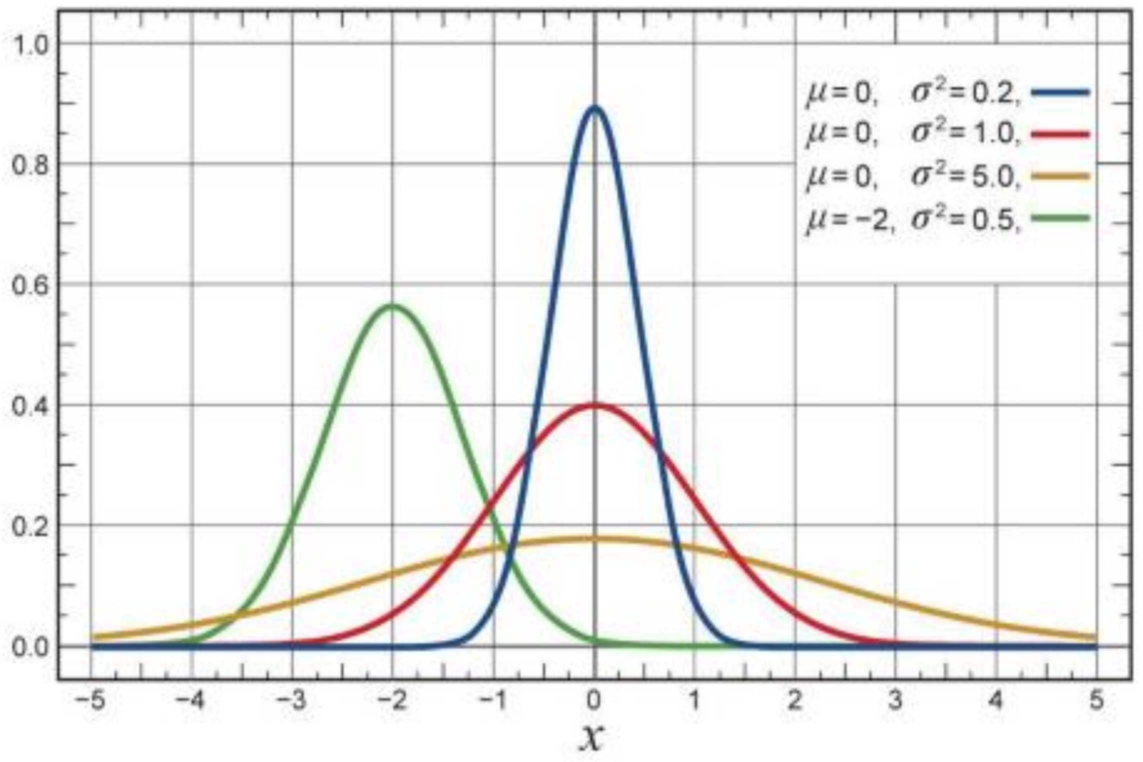
```
Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$
Input: a differentiable state-value function parameterization $\hat{v}(s, \mathbf{w})$
Algorithm parameters: $\lambda^{\mathbf{w}} \in [0, 1]$, $\lambda^{\boldsymbol{\theta}} \in [0, 1]$, $\alpha^{\mathbf{w}} > 0$, $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\bar{R}} > 0$
Initialize $\bar{R} \in \mathbb{R}$ (e.g., to 0)
Initialize state-value weights $\mathbf{w} \in \mathbb{R}^d$ and policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ (e.g., to $\mathbf{0}$)
Initialize $S \in \square$ (e.g., to $s_0$)
$\mathbf{z}^{\mathbf{w}} \leftarrow \mathbf{0}$ ($d$-component eligibility trace vector)
$\mathbf{z}^{\boldsymbol{\theta}} \leftarrow \mathbf{0}$ ($d'$-component eligibility trace vector)
Loop forever (for each time step):
    $A \sim \pi(\cdot|S, \boldsymbol{\theta})$
    Take action $A$, observe $S'$, $R$
    $\delta \leftarrow R - \bar{R} + \hat{v}(S', \mathbf{w}) - \hat{v}(S, \mathbf{w})$
    $\bar{R} \leftarrow \bar{R} + \alpha^{\bar{R}} \delta$
    $\mathbf{z}^{\mathbf{w}} \leftarrow \lambda^{\mathbf{w}} \mathbf{z}^{\mathbf{w}} + \nabla \hat{v}(S, \mathbf{w})$
    $\mathbf{z}^{\boldsymbol{\theta}} \leftarrow \lambda^{\boldsymbol{\theta}} \mathbf{z}^{\boldsymbol{\theta}} + \nabla \ln \pi(A|S, \boldsymbol{\theta})$
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}} \delta \mathbf{z}^{\mathbf{w}}$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}} \delta \mathbf{z}^{\boldsymbol{\theta}}$
    $S \leftarrow S'$
```

Policy Parameterization for Continuous Action

$$\pi(a|s,\boldsymbol{\theta}) \doteq \frac{1}{\sigma(s,\boldsymbol{\theta})\sqrt{2\pi}} \exp\left(-\frac{(a-\mu(s,\boldsymbol{\theta}))^2}{2\sigma(s,\boldsymbol{\theta})^2}\right)$$

$$\mu(s,\boldsymbol{\theta}) \doteq \boldsymbol{\theta}_\mu{}^\top \mathbf{x}_\mu(s) \quad \text{and} \quad \sigma(s,\boldsymbol{\theta}) \doteq \exp\left(\boldsymbol{\theta}_\sigma{}^\top \mathbf{x}_\sigma(s)\right)$$

$$\nabla_\theta \log \pi_\theta(s,a) = \frac{(a-\mu(s))\phi(s)}{\sigma^2}$$



Reference

1. David Silver, Lecture 7: Policy Gradient
2. Chapter 13, Richard S. Sutton and Andrew G. Barto, “Reinforcement Learning: An Introduction,” 2nd edition, Nov. 2018